Optimization of a Tunable Piezoelectric Harvester Applied to Multimodal Structures

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Abstract — The field of energy harvesting experienced a constant growth in the last years, due to the possibility of developing standing-alone wireless portable devices with extended life. In this context, piezoelectric materials appear to be particularly effective for the development of harvesters able to scavenge energy from ambient vibrations. In this paper a piezoactuated cantilever beam used for energy harvesting purposes is considered, extracting energy from a vibration source applied at the clamped boundary. The piezoelectric dimensions and position are optimized in order to maximize the coupling on the vibration modes of interest. An electric circuit containing a resistor and an inductor, connected to the piezoelectric electrodes, is optimized, for extracting the maximum electric power for any frequency of the vibration source, accounting for several vibration modes of the structure. The inductance is used to compensate the presence of a mistuning between the vibration source and the cantilever resonance frequencies. Proposed analysis shows that a single inductance is much effective when the harvester can be treated essentially as a single-degree-of-freedom structure. For harvesters with multiple degrees-of-freedom a single inductance can perform only a trade-off compensation of the mistuning between the various modes.

Keywords — Energy harvesting; Piezoactuated beams; Tuning strategies; Structural vibrations; Resonant circuits.

I. INTRODUCTION

The possibility of harvesting ambient energy to convert into electrical one has aroused in the last years the interest of many researchers. As a matter of fact, due to the constant growth and development of wireless sensors and wearable electronics (generally consuming a larger amount of power than in the past as a result of their increased functionalities and computing capabilities) the possibility of substituting classical electrochemical batteries with renewable energy sources has become a challenge [1]. Several environmental energy sources are commonly available, such as solar, thermic, aeolic energy. Nevertheless, largely diffused energy sources are associated to environmental vibrations, which are commonly and continuously present everywhere [2]. Among different smart materials available for energy scavenging from ambient vibrations, piezoelectric materials occupy a leading role and have recently seen an increasing rise in the use for such a purpose [3]. They have high electromechanical coupling, exhibit a prompt and linear response on a wide frequency range, have low specific weight, and are easily integrable in various structures including MEMS (namely, Micro-Electro Mechanical Systems) [4]. The most easy and studied harvester configuration is a cantilever beam with a piezoelectric device bonded near the clamped edge. The cantilever, rigidly connected to a vibrating structure representing the source of ambient vibrations, is generally considered as a single mode resonator inducing a periodic strain into the piezoelectric device. Due to the direct piezoelectric effect, a periodic electric field arises in the piezoelectric thickness, generating a difference of potential between the piezoelectric electrodes. The electric energy produced by the harvester can be used for feeding an electric load connected to the piezoelectric electrodes, or for recharging batteries. In any case, the amount of the harvested energy is strongly dependent on the electric impedance characterizing the external circuit.
It is worth observing that an immediate consequence of the energy scavenging process is an energy subtraction from the vibrating harvester, i.e., the piezoelectric device exerts a damping action on the vibrating piezoelectric actuator [5], thus reducing its harvesting capabilities. As a consequence, the optimization of the harvester parameters must take into account the complete electromechanical problem [6].

Many recent studies have been devoted to the modeling of piezoelectric energy harvesters, focusing on the optimal choice of the electric impedance to be connected to the piezoelectric electrodes, in order to maximize the scavenged energy. The simplest model considers the harvester as a single-degree-of-freedom system, mounted on a single-harmonic vibration source. In this case and considering a simple resistance connected to the piezoelectric electrodes, it is easy to prove the existence of two optimal values of resistance, yielding a maximum extracted power when the frequency of the vibration source is equal to the harvester eigenfrequency at short and open piezoelectric electrodes, respectively [7].

In order to widen the effective frequency band to the harvesting device several method have been proposed in the literature. For example the use of mechanically tunable devices [8], magnetically tunable ones [9], semiactive harvesting circuits [10], the exploitation of suitably introduced nonlinearities [11-13] or the use of dynamic magnifiers [14]. Also the addition of suitable tunable reactive impedances to the harvesting resistance can be sought in order to enlarge the frequency band of the harvester. In [15] the addition of a capacitance in series to the resistance is studied; a positive capacity does not lead to any remarkable improvement, whereas the addition of a negative capacity is proven to yield a constant value of extracted power at any frequency of the vibration source. Of course, a negative capacitance is an active component, requiring an external electric feed and possibly presenting instability problems. A constant extracted power, with respect to the vibration source frequency, is shown to be achievable by adding an inductance in series to the electric load resistance [16]. The inductance allows to compensate the mistuning between the vibration source frequency and the resonance frequency of the harvester device, by introducing a tunable second resonant mode.

A similar principle was adopted in [6] in the case of a tuned mass harvester and in [17] for a magnetic induction harvester. In these cited examples single-degree-of-freedom vibrating structures have been considered in the analysis and the optimization. To the author’s best knowledge very few examples have been devoted to the case of multi-mode structures.

In this paper the modeling and the optimization of a multi-mode piezoelectric energy harvester is addressed. A modal reduction is employed in order to obtain a discrete model of the vibrating system, taking into account several structural eigenmodes.

The problem of the optimal along-the-beam position and length of the piezoelectric device is discussed, in such a way as to maximize the piezoelectric coupling coefficient relevant to the considered structural eigenmodes.

The solution proposed in [16] is studied by accounting for more eigenmodes in the harvester structural model, highlighting a different behavior than the single-mode case. In detail, the addition of a single inductance enables to extract a maximum electric power only around the involved structural eigenfrequencies. For vibration frequencies between two structural eigenfrequencies, the inductance is only able to perform a trade-off compensation of the mistuning, not leading to a constant power extraction. A solution aimed at improving the power extracted in multi resonant energy harvester could consist in more parallel resonant branches in the electric circuit applied to the piezoelectric electrodes.

II. ELECTROMECHANICAL MODEL OF A PIEZOELECTRIC BEAM HARVESTER

In this paper a simple device for the energy harvesting from environmental mechanical vibrations is addressed. In detail, we consider an elastic beam perfectly clamped at one end to a vibrating rigid basement and equipped with a thin piezoelectric layer. The beam has length \( \ell \) and has a rectangular cross-section with thickness \( t \), width \( b \), and flexural moment of inertia \( J=br^3/12 \). It is comprised by a linearly elastic isotropic material with Young modulus \( E \), and mass per unit length \( \rho \). A Cartesian frame \( (O, x, y, z) \) is introduced, with \( z \in (0, \ell) \) spanning the beam axis.

![Figure 1. Piezoactuated cantilever harvester: cantilever beam rigidly fixed to a vibrating base, piezoelectric actuator and RL external circuit.](image)
Figure 2. Schematic representation of the bimodal reduced model.

The dynamical behaviour of the present electro-mechanical continuous harvester system, excited by the base vibration, can be modeled by means of a discrete formulation and analyzed by employing standard modal analysis techniques. In this paper we address two cases. In the first one the circuit parameters are optimized to maximize the power extracted by considering the system as a single-degree-of-freedom structure vibrating according to its first eigenmode. In the second case, we optimize the system considering the presence of the first two structural eigenmodes. For the structure under consideration such two modes correspond to the first two flexural modes in the plane (y, z). Denoting as \( f \) the first time derivative of \( f \), the equations governing the dynamical response of the system when the first two eigenmodes are considered, result in:

\[
\begin{align*}
\dot{m}_1 \ddot{\xi}_1 + c_1 \dot{\xi}_1 + k_{11}^{mm} \xi_1 + k_{12}^{me} \xi_2 + k_{15}^{me} \nu = f_1 \\
\dot{m}_2 \ddot{\xi}_2 + c_2 \dot{\xi}_2 + k_{22}^{mm} \xi_2 + k_{25}^{me} \xi_1 + k_{26}^{me} \nu = f_2 \\
k_{15}^{me} \xi_1 + k_{25}^{me} \xi_2 + k_{55}^{me} \nu + q = 0 \\
L \dot{q} + R \dot{q} + \nu = 0
\end{align*}
\]  

(2)

where \( \nu \) is the electric potential difference (voltage) across the two piezoelectric electrodes, \( q \) is the electric charge stored at the electrodes, and \(-k_{55}^{me}C_p\) is the clamped electric capacity of the actuator. Moreover, referring to the \( i \)-th mode \((i=1,2)\), \( \xi_i(t) \) denotes the modal coordinate corresponding to the eigenmode \( \psi_i(z) \), \( m_i = \int_0^\ell \rho' \psi_i^2(z) \, dz = 1 \) kg is the normalized modal mass, \( k_{ii}^{me} \) is the short-circuit modal mechanical stiffness, \( c_i \) is the modal structural damping, \( k_{ij}^{me} \) is the modal coupling stiffness, and \( f_i \) is the modal force induced by the base vibration. The base is considered harmonically vibrating with circular frequency \( \omega \), i.e. \( y_b(t) = Y_0 e^{i\omega t} \), \( j = (-1)^{1/2} \) being the imaginary unit. Therefore, the modal forces read as:

\[
f_1 = \omega^2 Y_0(t) m, \quad f_2 = \dot{A} f_1(t)
\]

(3)

with

\[
m = \int_0^\ell \rho' \psi_1^2(z) \, dz, \quad A = \frac{1}{m} \int_0^\ell \rho' \psi_2^2(z) \, dz
\]

(4)

and all time-dependent unknown functions (namely, \( \xi_i \), \( q \), \( \nu \)) are themselves harmonic in time, with circular frequency \( \omega \). The displacement field in the beam-like device is represented according to the Euler-Bernoulli model. Accordingly, the closed-circuit stiffness properties of the harvester system can be determined referring to a homogenized flexural stiffness \( (EJ) \), generally depending on the beam Young modulus \( E \), on the reduced elastic modulus along the z-axis of the piezoelectric material \( E_p \), on the thicknesses \( t \) and \( t_p \), as well as on the actuator position. The two-modes-based model is sketched out in Figure 2, wherein the physical meaning of previous equations is highlighted. In
particular, Eqs. (21) and (22) are the force balance on the masses $m_i$. Eq. (21) is the electric-charge balance at the piezoelectric surfaces, and Eq. (22) is the Kirchhoff equation of the external circuit. When the first mode is considered, Eq. (21) has not to be considered and $k_{11}^{m_{e}}=0$. The clamped electric capacity $C_p$ and the coupling stiffness $k_{i1}^{m_{e}}$ of the $i$-th mode are expressed as:

$$C_p = -k_{i1}^{m_{e}} = \varepsilon_{33} \frac{b l_p}{t_p}$$

$$k_{i1}^{m_{e}} = \frac{t_p}{2} \varepsilon_{31} b [\psi'(z_i) - \psi'(z_0)]$$

where $\psi = d\psi/dz$, and the material constants $\varepsilon_{33}$ and $\varepsilon_{31}$ indicate the reduced clamped permittivity in the along-the-thickness direction and the reduced piezoelectric coupling coefficient, respectively [18]. The main mechanical and geometrical properties employed in this work are summarized in Table 1-2.

<table>
<thead>
<tr>
<th>Table 1. Mechanical and Geometrical Properties of the Beam</th>
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<tbody>
<tr>
<td>$\rho$ [kg/m]</td>
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<td>0.471</td>
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<tr>
<th>Table 2. Mechanical and Geometrical Properties of the Piezoelectric Patch</th>
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<td>$\rho_p$ [kg/m$^3$]</td>
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<td>3.56·10$^{-3}$</td>
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By combining Eqs. (21) and (22), for eliminating $v$, and by employing Fourier synthesis, the frequency-dependent governing equations reduces to:

$$\begin{cases}
D_1 \ddot{\xi} + d \ddot{\eta} = a \\
\ddot{\eta} = c \cdot \xi
\end{cases}$$

where $\xi(\omega)$ denotes the Fourier transform of the function of time $g$, and the symbol "$\cdot\cdot\cdot$" indicates the inner product operator. Moreover, the 2x2 matrix $D=[D_{ij}]$ and vectors $\xi$, $d$, $a$ and $c$ are defined as:

$$D_{11} = -\Omega^2 + 2j \zeta_{1} \Omega + 1 + \kappa_1^2, \quad D_{12} = D_{21} = \kappa_1 \kappa_2 \delta,$$

$$D_{22} = -\Omega^2 + 2j \zeta_{2} \Omega \delta + \delta^2 (1 + \kappa_2^2),$$

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \quad d = \begin{pmatrix} \kappa_1^2 / k_{11}^{m_{e}} \\ \kappa_2^2 / k_{22}^{m_{e}} \end{pmatrix}, \quad a = \Omega^2 \tilde{m} \tilde{y} \begin{pmatrix} 1/m_1 \\ 1/m_2 \end{pmatrix},$$

and then $\kappa_1^{m_{e}}=20.60$ N/mm, $\kappa_2=0.1248$. The optimal values of resistance $R$ and inductance $L$ can be determined maximizing the modulus of the harvested power $P$. In detail, solving Eqs. (6) (by assuming $\xi_2 = 0$) with respect to the complex amplitudes $\xi_1$ and $\eta$, the following normalized moduli result:

$$\frac{\tilde{Q}}{m_1 \omega_0^2 \tilde{y}_p} = \frac{1}{m_1} \sqrt{\frac{\omega^2 \tilde{F}^2 + (\beta^2 \tilde{g}^2 - \Omega^2 \tilde{y}_p)}{\omega^2 \tilde{y}_2}}$$

$$\frac{\tilde{q}}{m_2 \omega_0^2 \tilde{y}_p} = \frac{1}{m_2} \sqrt{\frac{\kappa_1^{m_{e}}}{\kappa_2^{m_{e}}}}$$

where

$$\Lambda(\alpha, \beta) = \sqrt{\tilde{F} N_1^2 + N_2^2},$$

$$N_1 = \alpha (1 + \kappa_1^2) + 2 \zeta - (\alpha + 2 \beta \zeta) \Omega^2,$$

$$N_2 = \beta (1 + \kappa_2^2) + 2 \zeta - (\alpha + 2 \beta \zeta) \Omega^2.$$
\[ N_2 = 1 - (1 + \beta + 2 \alpha \zeta + \beta \kappa) \Omega^2 + \beta \lambda^2. \] (13)

Therefore, the normalized modulus \( \Pi \) of the harvested power \( P = \tilde{q} R \alpha^2 \) is:

\[
\Pi = \left| \tilde{F} \right| \left( m_o \omega_o f_b \right)^2 = \frac{1}{m_1} \frac{\kappa_{kk}^{mm}}{\omega^2 \Lambda^2} \Omega^2. \] (14)

Accordingly, since \( P = R \gamma^2 \) and \( \sigma = R \Gamma \), where \( \Gamma \) is the electric current, the normalized moduli of the current and voltage amplitudes result in:

\[
\frac{\tilde{I}}{m_o \omega_o f_b} = \frac{1}{m_1} \frac{\kappa_{kk}^{mm}}{\omega^2 \Lambda^2} \Omega \] (15)

\[
\frac{\tilde{V}}{m_o \omega_o f_b} = \frac{1}{m_1} \frac{\kappa_{kk}^{mm}}{\omega^2 \Lambda^2} \sqrt{\frac{\kappa_{kk}^{mm}}{\gamma^2}} \Omega \] (16)

By imposing stationary conditions of \( \tilde{P} \) with respect to \( \alpha \) and \( \beta \), the corresponding optimum values result in:

\[
\alpha_{opt} = \frac{2 \gamma^2}{(1 + \kappa^2 - \eta^2 + (2 \beta \Omega)^2)} \] (17)

\[
\beta_{opt} = \frac{1}{\Omega} \frac{\{1 + \kappa^2 - \eta^2\} (1 - \Omega^2 + (2 \beta \Omega)^2)}{(1 + \kappa^2 - \eta^2 + (2 \beta \Omega)^2)} \] (18)

from which, by employing Eqs. (8) the optimum values of both \( R \) and \( L \) can be derived, as well as the optimal normalized harvested power \( \Pi \) immediately results in:

\[
\Pi_{opt} = \left| \tilde{F} \right| \left( m_o \omega_o f_b \right)^2 = \frac{1}{a_o} \frac{1}{m_1} \frac{k_{kk}^{mm}}{\omega^2 \Lambda^2} \] (19)

It is worth remarking that, when both resistance and inductance are tuned on their optimal values, which are independent each other, the optimal power harvested from the base vibration is constant with respect to the base frequency, fully in agreement with [16]. This occurrence is not verified when the inductance \( L \) is assumed to be zero and only a resistor component is considered in the external electric circuit. In such a case, it is possible to show by analogous considerations that the optimal value for the resistance parameter \( \alpha \) is:

\[
\alpha_{opt} \Big|_{\beta = 0} = \frac{1}{\Omega} \sqrt{\frac{(1 - \Omega^2)^2 + (2 \beta \Omega)^2}{(1 + \kappa^2 - \eta^2)^2 + (2 \beta \Omega)^2}} \] (20)

and as a consequence, the optimal harvested power evaluated by Eq. (14) under the constraint \( \beta = 0 \) is strongly dependent on \( \Omega \). In detail, the normalized optimal power harvested by considering only the first structural eigenmode and including or not the inductor component is plotted in Figure 3 versus \( \Omega \) and for different values of the damping ratio \( \zeta \). With only a resistive electric load, optimally chosen according to Eq. (20), the maximum power is extracted at two frequencies, corresponding to the structural eigenfrequency at open- and shorted electrodes, respectively. These two frequency points coalesce into a single point when the structural damping is increased. By adding an inductor in series with the resistor and optimally chosen according to Eqs. (18), a perfect compensation of the mistuning between vibration source and structural eigenfrequency is obtained, yielding an extracted power constant in frequency. Furthermore, the analysis of Eq. (19) highlights as the maximum of the energy harvested by the vibrating base reduces when increments in the structural damping and/or stiffness are accounted for.

![Figure 3. Single-mode harvester. Optimal normalized harvested power \( \Pi_{opt} \) vs. the dimensionless base frequency \( \Omega \), for different values of the modal structural damping \( \zeta \). Comparison between harvesting performance obtained by purely resistive external circuit (R) and including inductor (RL).](image)

Figure 4 shows, for different levels of structural damping and referring to the RL circuit, the optimal values of the dimensionless resistance \( \alpha \) and of the dimensionless inductance \( \beta \) versus \( \Omega \). The presence of structural damping greatly affects the optimal value of the resistance, exhibiting a maximum in correspondence of the structural eigenfrequency, whereas it does not influence the optimal value of the inductance. This latter follows an hyperbolic behaviour with respect to the frequency, accomplishing to the tuning condition \( 1 / \sqrt{L \rho C_o} \equiv \omega \) on the vibration source frequency, almost everywhere except around the structural eigenfrequency.

IV. OPTIMIZATION OF THE BIMODAL HARVESTER

Assuming that the device harvests power from the oscillating base by exciting two modes of the structure and in agreement with the previous considerations, the optimal position of the piezoelectric layer is determined by solving the following problem:
Find the interval \((z_0, z_1) \subset (0, \ell)\) that makes maximum the minimum value between the electro-mechanical coupling coefficients \(k_i\) (with \(i = 1, 2\)), that is
\[
\max_{(z_0, z_1) \subset (0, \ell)} \left( \min \left[ k_1, k_2 \right] \right) = \left( \min \left[ k_1, k_2 \right] \right) (21)
\]

Since the shape of the first two modes of a cantilever beam, the optimal position of the actuator is expected near the clamped boundary and with a length small in comparison with \(\ell\). Accordingly, in such a case it is possible to prove that the piezoelectric layer contributes negligibly to the stiffness of the overall structure, so that modal properties of the system can be approximated with the modal properties of the beam only, that is \(\rho' \approx \rho\) and \((EJ)' \approx EJ\). Therefore, for the beam herein considered, the values of the closed-circuit mechanical stiffnesses result equal to \(k_{1mm} = 15.87\) N/mm and \(k_{2mm} = 623.44\) N/mm. Moreover, the problem (21) is satisfied by \(z_0 = 0\), \(z_1/\ell = \ell_1/\ell = 0.14\), and corresponding values of the electro-mechanical coupling parameters are \(k_1 = 0.1273\), \(k_2 = 0.0941\). By solving Eqs. (6) the modal amplitude vector \(\tilde{\xi}\) and the charge amplitude \(\tilde{q}\) result in:
\[
\tilde{\xi} = \left[ D + d \otimes c \right]^{-1} a, \quad \tilde{q} = c \cdot \left[ D + d \otimes c \right]^{-1} a
\]
where the symbol “\(\otimes\)” indicates the tensor product operator. Accordingly, the normalized modulus of the harvested power \(\Pi\) can be put in the form
\[
\Pi = \frac{\left| \tilde{q} \right|^2}{m^2 \omega^2 \pi^2 \eta Y} = \frac{\alpha k_1 k_2 \delta^2}{k_{1mm}} \Omega_1 (\omega) (23)
\]

A. Absence of structural damping

On the first we consider the case relevant to a zero structural damping on each mode, namely \(\zeta_i = \zeta_i = 0\). In this case, the stationary conditions with respect to \(\alpha\) and \(\beta\) of the harvested power expressed by Eq. (23) reveals that \(\Pi\) is maximum in correspondence of the following optimum value of the dimensionless inductance parameter \(\beta\)
\[
\beta_{opt} = \frac{\delta^2 - \delta^2 (1 + \delta^2) + \delta^2 + \kappa_1 (\delta^2 - \Omega^2) + \kappa_2^2 (1 - \Omega^2)^2}{\alpha (\delta - \Omega^2)} (24)
\]

where values of the dimensionless resistance parameter \(\alpha\) that maximize the power do not exist. It should be noted that \(\beta_{opt}\) does not depend on \(\alpha\) as well as on the modal ratio \(A\). Under the constraint \(\beta = \beta_{opt}\), the optimal harvested power results in:
\[
\Pi_{opt} = \frac{\left| \Delta k \delta^2 (\delta^2 - 1) + \delta^2 (\kappa_1 \delta - \Omega^2) + \kappa_2^2 (1 - \Omega^2)^2 \right|}{\alpha \delta^2 (\delta - \Omega^2)} (25)
\]
and, since \(\alpha\) must be positive, the optimal power harvested by the bimodal device is described by an hyperbolic function with respect to \(\alpha\), that is the power tends to infinity when the resistance \(R\) tends to zero, as well as the power runs out to zero when \(R\) assumes high values. As a matter of fact, this singular behaviour of the optimal resistance, which is a consequence of the assumption of zero structural damping, is not realistic. Figure 5 shows, for the case under consideration, the behaviour of the optimal inductance, revealing essentially the same trend as in the case of only one single structural eigenmode (see Figure 4). In the case of the bimodal harvester, two local deviations from the tuning behavior \(1/\sqrt{L_C} \approx \omega\) appear around the two structural eigenfrequencies. Figure 6 reports the corresponding optimal extracted power versus the dimensionless base frequency. It can be noted that, considering two eigenmodes \(\Pi_{opt}\) is no longer constant in frequency, but exhibits two peaks in correspondence of the two structural resonances.

![Figure 4](image-url) 

Figure 4. Single-mode harvester based on the RL external circuit. Optimal values of the dimensionless resistance \(\alpha\) (on the left) and of the dimensionless inductance \(\beta\) (on the right) vs. the dimensionless base frequency \(\Omega\) and for different values of the structural damping \(\zeta_i\).
B. Presence of the structural damping

When structural damping is accounted for, it is possible to determine optimal values of both $\alpha$ and $\beta$ that maximize the harvested power by two modes. The analytical relationships that characterize such values are herein omitted for the sake of compactness. Nevertheless, optimal dimensionless resistance $\alpha$ when structural damping is accounted for is plotted versus dimensionless base frequency in Figure 7. It exhibits two maxima in correspondence of the two structural eigenfrequencies, recalling a behaviour similar with respect to the optimal resistance in the single-mode case (see left hand side of Figure 4). As in the single-mode case, the behaviour of the optimal inductance, reported in Figure 8 referring to the dimensionless parameter $\beta$, is not significantly affected by the presence of structural damping, resulting practically coincident to the one obtained when damping effects are disregarded (see Figure 5). Considering the electrical components characterized by their optimal values, it is possible to prove that the optimal harvested power is strongly dependent on the frequency ratio $\delta$ and on the modal ratio $A$ (see Eqs. (7) and (23)). In detail, Figure 9 highlights the influence on $P_{\text{opt}}$ of the ratio $\delta$ between the first two natural frequencies. It can be observed that for large values of $\delta$, implying a large separation between the two structural eigenfrequencies, the optimal extracted power recovers the ideal behaviour obtained in the single-mode case, which is independent on the vibration source frequency. Nevertheless, it has to be remarked that, for the structural device herein considered the frequency ratio results $\delta = 6.268$. Accordingly, the optimal harvested power evaluated considering two modes is comparable with the one obtained considering a single-mode harvester only for a small interval of dimensionless base frequency, close to the structural eigenfrequencies. Finally, Figure 10 shows the
dependency of $\Pi_{\text{opt}}$ on the modal ratio $A$. It could be changed by considering different displacement boundary conditions for the harvester (in the case under consideration it results $A = 0.55$) and, as the analysis of the proposed results highlights, it mainly affects the value of the harvested power in ranges of the base frequency far from the structural eigenfrequencies. In detail, for high values of $\Omega$, an increase in $A$ induces an increase in $\Pi_{\text{opt}}$.

![Figure 9](image_url)  
**Figure 9.** Bimodal harvester based on the RL external circuit and considering the structural damping. Optimal normalized harvested power $\Pi_{\text{opt}}$ vs. the dimensionless base frequency $\Omega$ for different values of the frequency ratio $\delta$ ($\zeta_1 = 0.002$, $\zeta_2 = 0.003$, $\delta = 6.268$, $A = 0.55$).

![Figure 10](image_url)  
**Figure 10.** Bimodal harvester based on the RL external circuit and considering the structural damping. Optimal normalized harvested power $\Pi_{\text{opt}}$ vs. the dimensionless base frequency $\Omega$ for different values of the modal ratio $A$ ($\zeta_1 = 0.002$, $\zeta_2 = 0.003$, $\delta = 6.268$, $A = 0.55$).

V. CONCLUSIONS

A piezoelectric cantilever employed for energy scavenging from a vibration source has been studied in this paper. The source was considered as a single-harmonic acceleration applied at the cantilever basement. A discrete model of the coupled electromechanical system was derived, on the basis of a continuous model, by applying a modal reduction technique. Electric power was produced by the harvester by connecting either a resistive or a resistive-inductive electric circuit to the piezoelectric electrodes. An analytical optimization of the circuit parameters was developed, aimed at maximizing the extracted power, by taking into account in the analysis only a single or two structural eigenmodes. The dependence of the optimal electrical components with respect to the various parameters entering in the problem was shown and discussed. In particular, proposed analysis revealed that the presence of multiple structural eigenmodes partially compromises the capability of the inductance to compensate for a mistuning between the vibration source frequency and the structural eigenfrequencies. Future works will be devoted to analyze more complex electric circuits, containing various resonant parallel branches, which are expected to improve the device behaviour in the multimodal case. An extension of this analysis, with the goal of maximizing the extracted power in more realistic cases characterized by a vibration source containing several harmonics, is the object of a current work.

REFERENCES


